Photoinduced Deformations of Beams, Plates, and Films

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Photoresponsive solids such as nematic photoelastomers can undergo large deformations induced by light absorbed into rodlike molecules which bend and disrupt liquid crystal order. Significant variation of photoabsorption through the solid leads to nonuniform elastic deformations such as bending of beams and plates and pitting of layers. Such effects are also found in the presence of inhomogeneous thermal or swelling fields in solids or gels. We analyze the small deflection limit of these problems and show that beams made of these materials can have two elastically neutral planes, and that plates of these materials have a typical saddle shape. We also give a scaling analysis of the elasticity of photoinduced mounds and pits and speculate on their applications.

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Rubber formed from nematic liquid crystalline polymers has the simplest (uniaxial) orientational order associated with the aligned rodlike elements of their component polymers. It is characterized by a director $\hat{\mathbf{n}}$ characterizing the direction and an order parameter Q giving the extent of the order. Since rubber locally has fluidlike positional order and mobility, it is capable of large extensions. It almost flows, distortions being resisted only by the occasional cross-link between chains. The shear modulus is correspondingly low, $\mu \sim 10^4$–10^6 J/m^3 and deformations thus occur at constant volume (the change of which is penalized by a much higher energy scale), so that the Poisson ratio is $\nu = 1/2$.

Orientational order elongates chains, and since macroscopic shape and molecular extent are directly related in rubber, this order causes mechanical deformations. These deformations can be by as much as 400% when a rubber is taken through its thermal nematic-isotropic phase transition temperature where Q is either lost (on heating) or recovered (on cooling). This spectacular effect has been proposed as a possible route to mechanical actuation and artificial muscles [1,2]. A photoresponse of the same magnitude as thermal response can also be achieved [3]—the nematic order Q can be reduced optically rather than thermally by using nematic rods in the network polymers that have a photochromic (dye) center. A photon absorbed here induces an isomerization of the molecular structure, typically via a bend. This leads to a reduction in nematic order as when raising temperature and thus results in the same spontaneous strains. Indeed, since mechanical contractions in nematic elastomers depend only on the degree of order, one can directly map thermal and optical responses onto each other [3,4]. The molecular ground state can be recovered by either a thermal or a photostimulated back reaction and can thus reverse the mechanical strain. Similar, nonoptical effects can also occur in other materials such as drying and swelling gels and other noncrystalline solids [5], in thermally actuated bimetal strips [6], and in solids with nonequilibrium temperature gradients. Thus, although we will focus on the case of nematic photoelastomers, much of our analysis is applicable to other cases as well.

In this Letter we investigate the mechanical consequences of inhomogeneous illumination. For a free rubber beam or plate of thickness w, width W, and length L, linear absorption of photons incident from one side gives rise to an exponential attenuation of the optical intensity $I(x)$ with the distance penetrated x so that $I(x) = I(0) \times \exp\{-x/d\}$ and the attenuation length d characterizes the density of chromophores and the coupling to the optical field. Such elastic beams or plates will clearly bend as well as contract since there is relatively more contraction near the face at which the light enters. We explicitly connect the intensity and penetration of radiation with the curvature and with the position of neutral planes in the rubber, and show that the curvature can be tuned using the attenuation length so that it goes through a maximum. For long beams $L \gg W > w$, only one component of the curvature is important, see Fig. 1, while for plates and shells both competing curvatures can be important. However, as we will discuss, Gaussian curvature is soon lost in favor of large developable deformations.

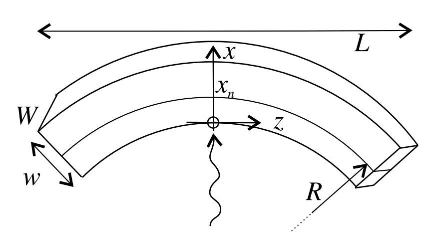

FIG. 1. A photobeam, initially straight and illuminated from below, contracts more where the light is less attenuated. The curvature induced is R. A neutral surface is at x_n.

Finally, we consider the shining of spots of light on a film of photoelastomer anchored to a substrate giving rise to localized deformations in the form of pits and mounds and speculate on their applications.

Consider a beam of photorubber shown in Fig. 1 illuminated from $x < 0$ by light which strikes the lower surface at $x = 0$. We take the director along the z axis of the strip, the most commonly achievable monodomain configuration. The photostrain (or free strain) $\varepsilon^r_{zz}(x) < 0$ in the photostationary state (where the forward photoreaction is balanced by the back reactions) is that which a body would suffer were it *uniformly* illuminated with light of the intensity found at x. Suppressing the tensor indices zz on ε^r from now on, we write the corresponding lateral photoexpansions perpendicular to the director as $\varepsilon^r_{xx} = \varepsilon^r_{yy} = -\varepsilon^r/2 > 0$ since rubber deforms incompressibly. Then the internal elastic response, e.g., internal stress, is due to the deformations relative to this new local "natural" state of the material in the body. In general, the actual strain minimizing the energy of the body is very different from $\varepsilon^r(x)$ (except at the neutral surfaces) since this is a nonlinear nonlocal problem involving finite isochoric deformations. However, in considering small strains as we do here, we ignore the distinction between the current and reference configurations, so that the axial elastic strain is then approximately proportional to the intensity, i.e., $\varepsilon^r(x) = -C\exp\{-x/d\}$, where C is a constant depending on temperature. Thus $\varepsilon^r(0) = -C$, and $\varepsilon^r(\infty) \to 0$ for thick beams, when $w \gg d$. Classical isotropic homogeneous beams have a single neutral plane, at $x_n = w/2$, which is unstrained. On one side of this plane the beam is stretched, while on the other side it is compressed. In photobeams we take the first neutral plane to be at a location x_n where there is no mechanically induced strain relative to the natural photocontracted state. On uniform illumination over $x = 0$, the beam bends into a circular arc of radius R; the z strain *relative* to the natural, irradiated state at x_n is $(x - x_n)/R$. In general, the beam is also curved in a perpendicular direction, leading to a saddle-shaped beam. However, for the case of long, slender beams considered previously, we can ignore this effect. This is because the combined effects of vanishing stresses on the surfaces with normals along x and y and the smallness of the x and y dimensions compared with the z dimension ($L \gg W > w$) implies that all other stresses are dominated by σ_{zz} and can thus be neglected to leading order. Then, the strain at x due to bending in just one direction is partially vitiated by the natural length of an infinitesimal element at x differing from that at x_n induced by the relative photostrain $\Delta(x - x_n) = [\varepsilon^r(x) - \varepsilon^r(x_n)] = Ce^{-x_n/d}(1 - e^{-(x-x_n)/d})$. The effective elastic strain e is then obtained by subtracting the relative photostrain $\Delta(x - x_n)$ from the bending strain $(x - x_n)/R$ associated with distance from the neutral surface in the presence of curvature, and yields

$$e(x) = \frac{x - x_n}{R} - \Delta(x - x_n). \tag{1}$$

The local elastic stress tensor has only one nonzero component, i.e., $\sigma_{zz}(x) = Ee(x)$, where E is the effective elastic modulus of the rubber in-plane stress [$E = Y/(1 - \nu^2)$, where Y is Young's modulus]. In the absence of any external forces and torques, the net forces and torques at any cross section must vanish. Therefore, for arbitrary $\varepsilon^r(x)$,

$$W\int_0^w Ee(x)dx = 0, \tag{2}$$

$$W\int_0^w xEe(x)dx = 0, \tag{3}$$

which are conditions determining the unknowns x_n and $1/R$. For weak absorption, $d \gg w$, light is hardly attenuated by passage through the photobeam; ε^r varies essentially linearly, $\Delta(x - x_n) \approx C(x - x_n)/d$. Then conditions (2) and (3) can be matched at all planes

$$\left(\frac{1}{R} - \frac{C}{d}\right)\left(\frac{1}{2}w^2 - wx_n\right) = 0,$$
$$\left(\frac{1}{R} - \frac{C}{d}\right)\left(\frac{1}{3}w^3 - \frac{1}{2}w^2 x_n\right) = 0 \tag{4}$$

by choosing $1/R = C/d$. Therefore every plane is neutral and no part of the beam is strained from its natural, irradiated state, so that no internal stresses are set up. The mean contraction of the beam is $\varepsilon^r(w/2)$, the shrinkage at the midpoint. Stronger optical absorption across the elastic beam implies that the variation of ε^r through the thickness is no longer linear, whereas strain due to bending remains so for simple geometric reasons. The bent beam then has internal strains $e(x)$ with respect to its natural irradiated state; there are extensions in the region $(0, x_n)$ where the photocontraction is large, a neutral plane at x_n (forward of the midplane) after which there is compression, and then a second neutral surface after which there is again extension. Thus the distribution of stresses is reversed with respect to classical beams and is just as in bimetallic strips [6]. More than one neutral plane is required for the simultaneous satisfaction of the two conditions (2) and (3). For the current, exponential case of interest they can be explicitly integrated to yield the beam curvature and a condition for the neutral planes x_n:

$$\frac{w}{R} = 12C\frac{d}{w}\left[\left(\frac{d}{w} + \frac{1}{2}\right)e^{-w/d} - \frac{d}{w} + \frac{1}{2}\right] \tag{5}$$

$$\frac{x_n}{w} = \frac{1}{2} - \frac{w}{12d}\,\frac{e^{-x_n/d} - \frac{d}{w}(1 - e^{-w/d})}{(\frac{d}{w} + \frac{1}{2})e^{-w/d} - \frac{d}{w} + \frac{1}{2}}. \tag{6}$$

We see that the curvature is set by the parameters C and d which characterize the photoinduced strain $\Delta(x)$ and the attenuation length of the illumination. However, the locations of the neutral planes are not affected by C at all, and instead depend on the ratio of the attenuation length and the thickness of the beam d/w. Equation (5) shows that the reduced curvature $w/R \sim Cw/d$ for large d/w, while the first correction to the infinite number of neutral planes present when $d/w = \infty$ yields neutral surfaces at $x_n = \frac{1}{2}(1 \mp 1/\sqrt{3})w$. This is corroborated by plotting the location of the neutral planes and the curvature as a function of d/w as shown in Fig. 2

In Fig. 3 we show the corresponding scaled axial elastic strain $e(x)/C$ evaluated using (1) as a function of the dimensionless transverse coordinate x/w. Since the natural length of the irradiated state as well as the natural curvature are determined by the photostrain C the scale of these internal strains is also set by C. For strong absorption, $d/w \ll 1$ so that photocontraction is limited only to a thin skin near $x = 0$. The bulk of the beam resists contraction, so that the skin is itself under extension with respect to its new natural length. In Fig. 3 we see that for $d/w = 0.05$ the first neutral surface is at $x/w \approx 0.092$ and vanishes when $d/w \to 0$ as $x_n/w \sim (d/w)\ln(w/4d)$. In this limit, the second neutral surface tends to $x_n/w = 2/3$. For intermediate values of absorption, say $d/w = 0.5$, the main extensional region is not so tightly confined to the vicinity of $x = 0$. For even weaker absorption, say $d/w = 5$, the elastic strain is distributed more nearly symmetrically but is very small since the natural, photostrain varies almost linearly in the interval $x = 0$ to $x = 1$ and is thus nearly accommodated by simple bending leading to a strain $(x - x_n)/R$. When the absorption is so weak as to nearly vanish, the curvature also vanishes since ultimately one has uniform contraction. Thus we expect that the curvature reaches a maximum for some intermediate value of d/w; Eq. (5) shows that $d/w = 0.372$ leads to a maximum curvature $w/R \sim 0.84C$.

When the length and width of the beam are comparable, $L \sim W$, we have a plate and illumination can lead to a doubly curved shell: zz contractions (along L) must be accompanied by expansions in the two other directions to conserve volume. Since the yy expansion at the lower face is greater than that at the upper face, a curvature $1/R_{yy}$ in the y-x plane of opposite sign to that in Fig. 1 will initially result leading to a saddle-shaped shell with a negative Gaussian curvature. Unlike in the case of a beam, for a plate when $W \sim L$ we cannot neglect the in-plane stresses even in the limit of small deflections. The two conditions (2) and (3) are now replaced by four; two of which impose the condition of vanishing force resultants in the z and y directions, and the other two which impose the condition of vanishing moment resultants in the same two directions. Eliminating between these conditions shows that the neutral surfaces associated with the zz and yy stresses are coincident and that the y curvature is opposite in sign from the z curvature as expected for a saddle shape. Furthermore, $1/R_{yy}$ scales exactly as the $1/R_{zz}$, but with the change $C \to -\frac{1}{2}C$, as expected when volume conservation governs transverse expansions. For large deflections, the shapes are no longer saddlelike since deformation to a doubly curved surface with large Gaussian curvature causes substantial stretching — an energetically much more expensive deformation than bending. Instead, the plates deform almost inextensionally, i.e., without any stretching, almost everywhere except in the vicinity of certain boundary layers, whose discussion we defer to the future.

Finally, we consider shining light in spots of radius a onto the surface of a photoelastomer of thickness H glued onto a rigid substrate, Fig. 4. Then, any deformation of the material leads to shear strains generated by the anchoring. For simplicity, we assume that (a) the director does not rotate in this elastic analysis — either in response to the light or to shears, (b) the penetration depth $d \gg H$, so that the photostrain is uniform in z, and (c) the illumination causes a localized uniaxial elongational free strain along z with respect to the initial length by an amount ε^r and corresponding contractions of $-\varepsilon^r/2$ in the x and y directions.

We give a scaling analysis of the elastic response. If the actual zz-strain response with respect to the unilluminated state is u_{zz}, then with respect to the new natural length the elastic strain $e \sim u - \varepsilon^r$, where we

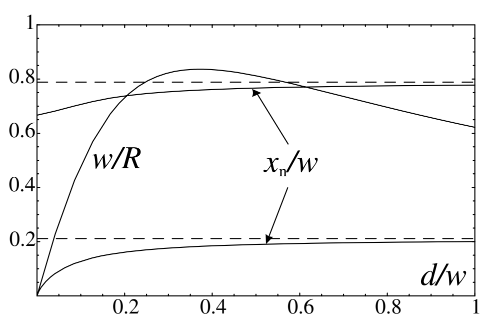

FIG. 2. The dimensionless beam curvature w/CR and the dimensionless location x_n/w of the neutral surfaces as functions of scaled penetration depth d/w. The x_n/w asymptotes $\frac{1}{2}(1 \mp 1/\sqrt{3})$ (dotted line) are rapidly approached as d/w increases.

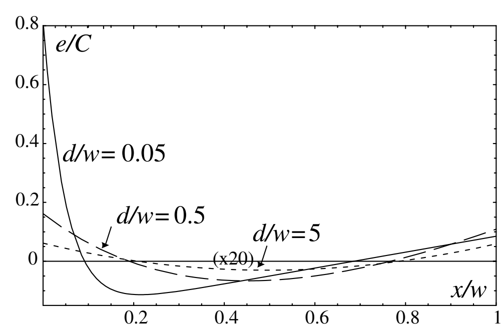

FIG. 3. The scaled elastic zz strain $e(x)/C$ as a function of x/w for different scaled penetration depths $d/w = 0.05$, 0.5, and 5, the last exaggerated $\times 20$ for visibility.

have suppressed indices as usual. Thus the stored elastic energy in the mound is $\sim E(u - \varepsilon^r)^2 a^2 H$. The material removed from the trough is equal in volume to that of the mound, i.e., $al\delta \sim a^2 uH$, where the healing distance $l \sim H$ (the scale over which three-dimensional effects decay) so that $\delta \sim au$. At the level of scaling, volume conservation then requires that the radial displacement of material points $u_r \sim -(a + H)u$. Since the bottom of the layer is glued to the substrate this leads to shear strains over a depth $\sim H$ of magnitude $u_r/H \sim u(a + H)/H$, and hence an energy $\sim Eu^2(a + H)^4/H$. Likewise in the rubber in the lip separating the trough and the mound there is a shear strain resulting from the variation of a z displacement u_z with r, that is, $u_{zr} \sim ua/H$ over a volume, $laH \sim aH^2$, yielding an energy $\sim Eu^2 a^3$. The material directly under the trough suffers a compressive strain $\sim \delta/H \sim ua/H$, again with energy $\sim Eu^2 a^3$. Lastly, there are compressive hoop strains $u_{\theta\theta} = u_r/r \sim -u/2$, the $\frac{1}{2}$ simply reminding us that these arise directly from volume conservation. The associated energy is $\sim Eu^2 a^2 H$. The total energy and minimizing strains are thus

$$\frac{F}{E} \sim a^2 H\left\{(u - \varepsilon^r)^2 + u^2\left[\frac{(1 + a/H)^4}{(a/H)^2} + 2\left(\frac{a}{H}\right) + 1\right]\right\},$$

$$u = \frac{\varepsilon^r}{1 + c(\frac{a}{H})}; \qquad c\left(\frac{a}{H}\right) = \left[\frac{(1 + a/H)^4}{(a/H)^2} + 2\left(\frac{a}{H}\right) + 1\right],$$

where the function $c(a/h)$ reflects the changing geometrical constraints as the spot changes size relative to the thickness of the rubber layer. For wide spots, the mound height above the background is small; that is, $uH \sim \varepsilon^r H^3/a^2$ as $a/H \gg 1$. Medium spots, $a/H \leq 1$, have a height $\sim H\varepsilon^r/n$ with n depending on the scaling and is a number between 1 and 10. For small spots the height is again small, $uH \sim \varepsilon^r a^2/H$ as $a/H \ll 1$. This analysis shows how thickness and spot size can be varied in order to tailor surface topography. Nematic elastomers that suffer uniaxial contraction and have the director normal to the surface will produce pits analogous to the mounds above. Such nematic photoelastomers with the director in plane (the most common geometry) will produce mounds of elliptical character. The scaling analysis should not be changed by this.

We conclude with a brief discussion of possible applications and future directions. An obvious application is actuation via optically induced bending [7,8]. Our demonstration that photobend displays a maximum, suggests how to choose materials and geometry to optimize an actuator. A recent interesting application is to an optical "swimmer," in effect a self-propelled, optically powered pump [8]. In polydomain rubbers [7], the curl direction is set purely by the polarization of the light, showing that the effect is purely optical and not the result of the effect of the heat generated by photon absorption. Recovery in these cases was via a back reaction stimulated by light of another color and can lead to rapid responses with rise times depending on the power, and decay times of about 10 ms [8]. Our analysis of pits and bumps in anchored and confined photoelastomers suggests applications to writable structures in microfluidics as valves, gates and reactor voids, tunable contact printing, switchable reflector elements in projective displays, etc., Our analysis has been limited to small deflection theory. Many important effects such as large strains (which are important in rubber), large amplitudes of bending, rates of photon absorption depending on the extent of photoresponse, shifts of the photostationary state by internal stresses, dynamics, and the application of external forces and torques remain to be studied.

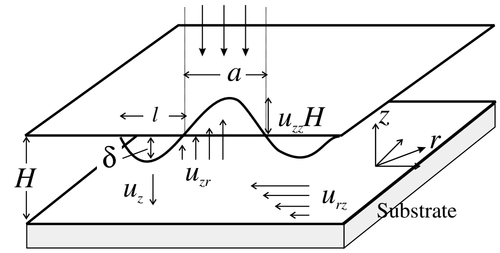

FIG. 4. A spot of light of radius a induces the thickness H of a photoelastomer, to increase by $u_{zz}H$. A trough of depth δ around the spot conserves volume. The healing length for distortions is $l \sim H$. Displacements u_r and u_z and resulting shears are sketched.

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