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Proc. R. Soc. A 2014 **470**, 20130604, published 12 March 2014

Supplementary data

["Data Supplement"](#)

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Cite this article: Lewicka M, Mahadevan L, Pakzad MR. 2014 Models for elastic shells with incompatible strains. *Proc. R. Soc. A* **470**: 20130604.

<http://dx.doi.org/10.1098/rspa.2013.0604>

Received: 9 September 2013

Accepted: 11 February 2014

Subject Areas:

analysis, mechanics, differential equations

Keywords:

non-Euclidean plates, nonlinear elasticity, Gamma convergence, calculus of variations

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Electronic supplementary material is available at <http://dx.doi.org/10.1098/rspa.2013.0604> or via <http://rspa.royalsocietypublishing.org>.

Models for elastic shells with incompatible strains

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The three-dimensional shapes of thin lamina, such as leaves, flowers, feathers, wings, etc., are driven by the differential strain induced by the relative growth. The growth takes place through variations in the Riemannian metric given on the thin sheet as a function of location in the central plane and also across its thickness. The shape is then a consequence of elastic energy minimization on the frustrated geometrical object. Here, we provide a rigorous derivation of the asymptotic theories for shapes of residually strained thin lamina with non-trivial curvatures, i.e. growing elastic shells in both the weakly and strongly curved regimes, generalizing earlier results for the growth of nominally flat plates. The different theories are distinguished by the scaling of the mid-surface curvature relative to the inverse thickness and growth strain, and also allow us to generalize the classical Föppl–von Kármán energy to theories of prestrained shallow shells.

1. Introduction

The physical basis for morphogenesis is now classical and elegantly presented in D'arcy Thompson's opus 'On growth and form' (p. 15) as follows: 'An organism is so complex a thing, and growth so complex a phenomenon, that for growth to be so uniform and constant in all the parts as to keep the whole shape unchanged would indeed be an unlikely and an unusual circumstance. Rates vary, proportions change, and the whole configuration alters accordingly'. From a mathematical and mechanical perspective, this reduces to a simple principle: differential growth in a body leads to residual strains that will generically result in changes

in the shape of a tissue, organ or body. Eventually, the growth patterns are expected to themselves be regulated by these strains, so that this principle might well be the basis for the physical self-organization of biological tissues. Recent interest in characterizing the morphogenesis of low-dimensional structures, such as filaments, laminae and their assemblies, is driven by the twin motivations of understanding the origin of shape in biological systems and the promise of mimicking them artificially [1–3]. The results lie at the interface of biology, physics and engineering, but they also have a deeply geometric character. Indeed the basic question of morphogenesis may be characterized in terms of a variation on a classical theme in differential geometry—that of embedding a shape with a given metric in a space of possibly different dimension [4,5]. However, the goal now is not only to state the conditions when it might be done (or not), but also to constructively determine the resulting shapes in terms of an appropriate mechanical theory.

While these issues arise in three-dimensional tissues, the combination of the separation of scales that arises naturally in slender structures and the constraints associated with the prescription of growth laws that are functions of space (and time) leads to the expectation that the resulting theories ought to be variants of classical elastic plate and shell theories such as the Föppl–von Kármán or the Donnell–Mushtari–Vlasov theories [6]. That this is the case has been shown for bodies that are initially flat and thin, i.e. elastic plates with no initial curvature, using analogies to thermoelasticity [7,8], perturbation analysis [3,9] and rigorous asymptotic analysis [10]. This follows a programme similar to the derivation of the equations for the nonlinear elasticity of thin plates and shells [11–16] and a linearized theory [17] for residually strained Kirchhoff plates [18]. However, most laminae are naturally curved in their strain-free configurations. Since even infinitesimal deformations of a curved shell will potentially violate isometry relative to its rest state, one expects that differential growth of such an object will likely lead to a variety of possible low-dimensional theories depending on the relative size of the metric changes imposed on the system. This multiplicity of asymptotic theories is of course presaged by a similar state of affairs for the derivation of a nonlinear theory of elastic shells [15,19].

We build on the discussion in [8,10,20] and present a rigorous derivation of a set of asymptotic theories for the shape of *residually strained* thin lamina with non-trivial curvatures, i.e. growing elastic shells. As our starting point, we use the observation that it is possible to change the shape of a lamina such as a blooming lily petal by driving it via excess growth of the margins relative to the interior, rather than via midrib deformations [21]. Previously, a thermoelastic analogy [7] suggested a natural generalization of the Donnell–Mushtari–Vlasov shell theory [6] to growing shells [20], proposed as a mathematical model for blooming activated by the initial (transverse) out-of-plane displacement v_0 of a petal's mid-surface. When $v_0 = 0$, equations (6.5) reduce to the prestrained von Kármán equations (6.3) proposed in [8]. These were rigorously derived in [10] from *non-Euclidean elasticity*, where the imposed three-dimensional prestrain is given via a Riemannian metric, whose components display the appropriate linear target stretching tensor ϵ_g (of order 2 in shell's thickness h), and the bending tensor κ_g (of order 1 in h , see (3.1)). This leads us to focus on a particular regime of scaling for the prestrain tensor (2.6) which corresponds, *a posteriori*, in all different regimes of shallowness studied here, to von Kármán-type theories.

It is pertinent to start with a few comments regarding this particular choice of the scaling regime. From a mathematical point of view, the von Kármán regime, where the nonlinear elastic energy per unit thickness scales like h^4 , usually corresponds to sub-linear theories, i.e. the first nonlinear theories which arise when the magnitude of forces or of prestrain allows the elastic lamina to cross the threshold of linear behaviour and lead to phenomena such as buckling. As these sublinear theories are also the least complicated among the nonlinear theories of plates and shells arising in the literature and are relevant for many applications, they are popular with engineers, physicists and applied mathematicians. Therefore, in the analysis of nonlinear shallow shell models with growth, it is reasonable to start with the von Kármán regime. By contrast, there are a number of technical challenges that must be addressed when deriving lower order nonlinear theories using Γ -convergence. Here we consider the first of a series that considers the various possible shell theories that result for various limiting cases of the growth strain, the boundary

loading, etc. In a forthcoming paper [22], we address a shallow shell model that arises in a forcing regime equivalent to the energy scaling h^β for $\beta < 4$, where, analogous to Friesecke *et al.* [19], technical obstacles regarding properties of the Sobolev solutions to the Monge–Ampère equations are addressed before establishing the corresponding Γ -limit result.

In §2, we formulate our main results, in terms of a scaling analysis that leads to the hierarchy of limiting models as a function of the various prestrain and shallowness regimes. In §3, we argue that for non-flat mid-surface S (with a natural out-plane displacement $v_0 \neq 0$), the variationally correct two-dimensional theory coincides with the extension of the classical *von Kármán energy* to shells, derived in [13]. In the special case $v_0 = 0$, the corresponding energy still reduces to the functional whose Euler–Lagrange equations are those derived for elastic plates in [8]. In §4, we discuss a new model valid when the radius of curvature of the mid-surface is relatively large compared with the thickness. This limit leads to a prestrained plate model which inherits the geometric structure of the shallow shell. In §5, we consider the case where the radius of curvature and the thickness are comparable in magnitude, and appropriately compatible with the order of the prestrain tensor. We show that equations for a growing elastic shell can be formally derived by pulling back the in-plane and out-of-plane growth tensors ϵ_g and κ_g , respectively, from *shallow shells* $(S_h)^h$ with reference mid-surface S_h given by the scaled out-of-plane displacement hv_0 , onto a flat reference configuration. Furthermore, we argue that this theory for growing elastic shells is also the Euler–Lagrange equation of the variational limit for three-dimensional nonlinear elastic energies on $(S_h)^h$. In §6, we discuss the model where the effects of shallowness are dominated by the growth-induced prestrain. In this case, the limiting energy is impervious to the influence of the shell geometry, but the effects of growth may not be neglected. This leads to the generalized von Kármán equations for a growing flat plate. In §7, we justify that under our prestrain or growth scaling assumptions, the derived models are the relevant ones when the boundaries are free and no external forces are present. Finally, in §8, we conclude with a discussion of the present results and prospects for the future. As the proofs of the theorems consist of tedious yet minor (though necessary) modifications of the arguments detailed in [10,13,14], we refer the interested reader to the electronic supplementary material, where they are given for completeness.

2. Preliminaries and scaling limits

Let $v_0 \in C^{1,1}(\bar{\Omega})$ be an out-of-plate displacement on an open, bounded subset $\Omega \subset \mathbb{R}^2$, associated with a family of surfaces, parametrized by $\gamma \in [0, 1]$

$$S_\gamma = \phi_\gamma(\Omega), \quad \text{where } \phi_\gamma(x) = (x, \gamma v_0(x)) \quad \forall x = (x_1, x_2) \in \Omega, \quad (2.1)$$

The unit normal vector to S_γ at $\phi_\gamma(x)$ is given by

$$\mathbf{n}^\gamma(x) = \frac{\partial_1 \phi_\gamma(x) \times \partial_2 \phi_\gamma(x)}{|\partial_1 \phi_\gamma(x) \times \partial_2 \phi_\gamma(x)|} = \frac{1}{\sqrt{1 + \gamma^2 |\nabla v_0|^2}} (-\gamma \partial_1 v_0(x), -\gamma \partial_2 v_0(x), 1) \quad \forall x \in \Omega.$$

For small $h > 0$, we now consider thin plates $\Omega^h = \Omega \times (-h/2, h/2)$ and three-dimensional shells $(S_\gamma)^h$

$$(S_\gamma)^h = \left\{ \tilde{\phi}_\gamma(x, x_3); x \in \Omega, x_3 \in \left(-\frac{h}{2}, \frac{h}{2} \right) \right\}, \quad (2.2)$$

where the extension $\tilde{\phi}_\gamma : \Omega^h \rightarrow \mathbb{R}^3$ of ϕ_γ on Ω^h in (2.1) is given by the following formula

$$\tilde{\phi}_\gamma(x, x_3) = \phi_\gamma(x) + x_3 \mathbf{n}^\gamma(x) \quad \forall (x, x_3) \in \Omega^h. \quad (2.3)$$

For an elastic body with the reference configuration $(S_\gamma)^h$, we assume that its elastic energy density $W: \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}_+$ is C^2 regular in a neighbourhood of $\text{SO}(3)$. Moreover, we assume that W satisfies the normalization, frame indifference and non-degeneracy conditions

$$\left. \begin{aligned} \exists c > 0 \quad \forall F \in \mathbb{R}^{3 \times 3} \quad \forall R \in \text{SO}(3) \quad W(R) = 0, \quad W(RF) = W(F), \\ \text{and} \quad W(F) \geq c \text{dist}^2(F, \text{SO}(3)), \end{aligned} \right\} \quad (2.4)$$

where $F = \nabla u$ is the deformation gradient relative to the reference configuration $(S_\gamma)^h$. For prestrained structures characterized by the Riemannian metric

$$p^h = (q^h)^T q^h \quad \text{on } (S_\gamma)^h,$$

the tensor $F = \nabla u$ is replaced by $F = \nabla u (q^h)^{-1}$, so that the thickness averaged elastic energy is given by

$$I^{\gamma, h}(u) = \frac{1}{h} \int_{(S_\gamma)^h} W(F) \, dz = \frac{1}{h} \int_{(S_\gamma)^h} W(\nabla u (q^h)^{-1}) \, dz, \quad \forall u \in W^{1,2}((S_\gamma)^h, \mathbb{R}^3). \quad (2.5)$$

Letting $\epsilon_g, \kappa_g: \bar{\Omega} \rightarrow \mathbb{R}^{3 \times 3}$ be two given smooth tensors, for each small h we define the growth tensors q^h on $(S_\gamma)^h$ by

$$q^h(\phi_\gamma(x) + x_3 \mathbf{n}^\gamma(x)) = \text{Id} + h^2 \epsilon_g(x) + hx_3 \kappa_g(x) \quad \forall (x, x_3) \in \Omega^h. \quad (2.6)$$

For a justification of the above model through interpreting q^h as the instantaneous growth tensor see [23]. The corresponding metric $p^h = (q^h)^T q^h$ on $(S_\gamma)^h$ is then

$$p^h(\phi_\gamma(x) + x_3 \mathbf{n}^\gamma(x)) = \text{Id} + 2h^2 \text{sym } \epsilon_g(x) + 2hx_3 \text{sym } \kappa_g(x) + \mathcal{O}(h^3).$$

An important part of our study focuses on the asymptotic behaviour in the limit of vanishing thickness $h \rightarrow 0$ of the variational models $I^{\gamma, h}$ in (2.5), when $\gamma = \gamma(h) = h^\alpha$ for a given exponent $0 \leq \alpha < +\infty$. The regime $\alpha > 0$ corresponds to the study of a *shallow shell*. However, we will identify *three distinct shallow shell limit models*, depending on the asymptotic behaviour of the ratio γ/h , which in our setting depends only on the value of α . This allows us to rigorously derive the Γ -limits: $\Gamma\text{-}\lim_{h \rightarrow 0} (1/h^4) I^{h^\alpha, h}$ and show that under suitable incompatibility conditions on the strain tensors ϵ_g or κ_g , the infimum of energies $I^{h^\alpha, h}$ scales like h^4 irrespective of the value of α . This justifies our choice of the energy scaling and lends credibility to limiting models as physically relevant in the corresponding scaling regimes.

To get a sense of our results, it is useful to summarize our analysis in terms of the Γ -limit of $(1/h^4) I^{h^\alpha, h}$, which can be identified as follows:

$$\Gamma\text{-}\lim_{h \rightarrow 0} \frac{1}{h^4} I^{h^\alpha, h} = \begin{cases} \mathcal{I}_4 & \text{if } \alpha = 0 \\ \mathcal{I}_4^\infty & \text{if } 0 < \alpha < 1 \\ \mathcal{I}_4^1 & \text{if } \alpha = 1 \\ \mathcal{I}_4^0 & \text{if } \alpha > 1. \end{cases} \quad (2.7)$$

The above four theories collapse into one and the same theory when $v_0 = 0$. Otherwise, we must deal with four distinct potential limits depending on the choice of parameters, in the following order:

Case 1. $\alpha = 0$. This corresponds to $\gamma = 1$ where the three-dimensional model is that of the prestrained nonlinear elastic shell of arbitrarily large curvature (no shallowness involved). We will show that the Γ -limit in this case leads to a prestrained von Kármán model \mathcal{I}_4 for the two-dimensional mid-surface S_1 . This will be described in a more general framework in §3.

Case 2. $0 < \alpha < 1$. This corresponds to the flat limit $\gamma \rightarrow 0$ when the energy can be conceived as a limit of the von Kármán models \mathcal{I}_4 for shallow shells S_γ . In other words, this limiting model

corresponds to the case when $\lim_{h \rightarrow 0} (\gamma(h)/h) = \infty$, and it can also be identified as

$$\mathcal{I}_4^\infty = \Gamma\text{-}\lim_{\gamma \rightarrow 0} \left(\Gamma\text{-}\lim_{h \rightarrow 0} \frac{1}{h^4} I^{\gamma,h} \right),$$

by choosing the distinguished sequence of limits, first as $h \rightarrow 0$ and then $\gamma \rightarrow 0$. In §4, we will see that \mathcal{I}_4^∞ is formulated for displacements of a plate but it inherits certain geometric properties of shallow shells S_γ , such as the first-order infinitesimal isometry constraint.

Case 3. $\alpha = 1$. This corresponds to the case $\lim_{h \rightarrow 0} \gamma(h)/h = 1$. The limit model \mathcal{I}_4^1 , derived in §5, is an unconstrained energy minimization, reflecting both the effect of shallowness and that of the prestrain. It corresponds to a simultaneous passing to the limit $(0, 0)$ of the pair (γ, h) in (2.5). The Euler–Lagrange equations (6.5) of \mathcal{I}_4^1 were suggested in [20] for the description of the deployment of petals during the blooming of a flower.

Case 4. $\alpha > 1$. Finally, the Γ -limit for all values of $\alpha > 1$, i.e. when $\lim_{h \rightarrow 0} (\gamma(h)/h) = 0$, coincides with the zero thickness limit of the degenerate case $\gamma = 0$, which is the prestrained plate von Kármán model, discussed in [10]. This limiting energy can be obtained by taking the consecutive limits

$$\mathcal{I}_4^0 = \Gamma\text{-}\lim_{h \rightarrow 0} \left(\Gamma\text{-}\lim_{\gamma \rightarrow 0} \frac{1}{h^4} I^{\gamma,h} \right).$$

3. The prestrained von Kármán energy for shells of arbitrary curvature: $\alpha = 0$

When the parameter $\alpha = 0$, the three-dimensional variational problem associated with (2.5) is reduced to the three-dimensional nonlinear elastic energy on the thin shell S_1^h , where S_1 is the graph of v_0 . It is useful to discuss this model in a more general framework. Let S be an arbitrary two-dimensional surface embedded in \mathbb{R}^3 , that is compact, connected, oriented and of class $C^{1,1}$. The boundary ∂S of S is assumed to be the union of finitely many (possibly none) Lipschitz continuous curves. We consider the family $\{S^h\}_{h>0}$ of thin shells of thickness h around S :

$$S^h = \left\{ z = x + t\mathbf{n}(x); x \in S, \frac{h}{2} < t < \frac{h}{2} \right\}, \quad 0 < h < h_0 \ll 1,$$

where we use the following notation: $\mathbf{n}(x)$ for the unit normal, $T_x S$ for the tangent space, and $\Pi(x) = \nabla \mathbf{n}(x)$ for the shape operator on S , at a given $x \in S$. The projection onto S along \mathbf{n} is denoted by π , so that $\pi(z) = x$ for all $z = x + t\mathbf{n}(x) \in S^h$, and we assume that $h \ll 1$ is small enough to have π well defined on each S^h .

The instantaneous growth of S^h is described, directly, by smooth tensors: $\epsilon_g, \kappa_g : \bar{S} \rightarrow \mathbb{R}^{3 \times 3}$, by

$$a^h = [a_{ij}^h] : \bar{S}^h \rightarrow \mathbb{R}^{3 \times 3} \quad \text{and} \quad a^h(x + t\mathbf{n}) = \text{Id} + h^2 \epsilon_g(x) + ht \kappa_g(x). \quad (3.1)$$

The growth tensor a^h is as in [8,10], now in a general non-flat geometry setting. Given the elastic energy density $W : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}_+$ as in (2.4), the thickness averaged elastic energy induced by the prestrain a^h is given by

$$I^h(u^h) = \frac{1}{h} \int_{S^h} W(\nabla u^h (a^h)^{-1}) \, dz, \quad \forall u^h \in W^{1,2}(S^h, \mathbb{R}^3). \quad (3.2)$$

Taking the asymptotic limit (the Γ -limit as $h \rightarrow 0$, see theorems 3.1 and 3.2) of the energies I^h (note that $I^h = I^{1,h}$ in the notation of (2.5)) then leads to the variationally correct model for weakly prestrained shells. It corresponds to the following nonlinear energy functional \mathcal{I}_4 acting on the admissible limiting pairs (V, B) :

$$\begin{aligned} \forall V \in \mathcal{V} \quad \forall B \in \mathcal{B} \quad \mathcal{I}_4(V, B) &= \frac{1}{2} \int_S Q_2 \left(x, B - \frac{1}{2} (A^2)_{\text{tan}} - (\text{sym } \epsilon_g)_{\text{tan}} \right) \\ &+ \frac{1}{24} \int_S Q_2(x, (\nabla(A\mathbf{n}) - A\Pi)_{\text{tan}} - (\text{sym } \kappa_g)_{\text{tan}}). \end{aligned} \quad (3.3)$$

Here, the space \mathcal{V} consists of the first-order infinitesimal isometries on S , defined by

$$\mathcal{V} = \{V \in W^{2,2}(S, \mathbb{R}^3); \tau \cdot \partial_\tau V(x) = 0 \text{ a.e. } x \in S \forall \tau \in T_x S\}, \quad (3.4)$$

that is those $W^{2,2}$ regular displacements V for whom the change of metric on S owing to the deformation $\text{id} + \epsilon V$ is of order ϵ^2 , as $\epsilon \rightarrow 0$. Furthermore, for a matrix field $A \in L^2(S, \mathbb{R}^{3 \times 3})$, let $A_{\text{tan}}(x)$ denote the tangential minor of A at $x \in S$, that is $[(A(x)\tau)\eta]_{\tau, \eta \in T_x S}$. The skew-symmetric gradient of V as in (3.4) then uniquely determines a $W^{1,2}$ matrix field $A : S \rightarrow SO(3)$ so that $\partial_\tau V(x) = A(x)\tau$ for all $\tau \in T_x S$. Hence, we equivalently write

$$\mathcal{V} = \{V \in W^{2,2}(S, \mathbb{R}^3); \exists A \in W^{1,2}(S, \mathbb{R}^{3 \times 3}) \text{ a.e. } x \in S \forall \tau \in T_x S \\ \partial_\tau V(x) = A(x)\tau \text{ and } A(x)^T = -A(x)\}.$$

For a plate, that is when $S \subset \mathbb{R}^2$, an equivalent analytic characterization for $V = (V^1, V^2, V^3) \in \mathcal{V}$ is given by $(V^1, V^2) = (-\omega y, \omega x) + (b_1, b_2)$, while the out-of-plane displacement $V^3 \in W^{2,2}(S, \mathbb{R})$ remains unconstrained.

The space \mathcal{B} in (3.3) consists of finite strains

$$\mathcal{B} = \left\{ L^2 - \lim_{\epsilon \rightarrow 0} \text{sym } \nabla w^\epsilon; w^\epsilon \in W^{1,2}(S, \mathbb{R}^3) \right\}, \quad (3.5)$$

which are all limits of symmetrized gradients of sequences of displacements on S . By $\text{sym } \nabla w(x)$ we mean here a bilinear form on $T_x S$ given by $(\text{sym } \nabla w(x)\tau)\eta = 1/2[(\partial_\tau w(x))\eta + (\partial_\eta w(x))\tau]$ for all $\tau, \eta \in T_x S$.

It follows (via Korn's inequality) that for a flat plate $S \subset \mathbb{R}^2$, the space \mathcal{B} consists precisely of symmetrized gradients of all the in-plane displacements: $\mathcal{B} = \{\text{sym } \nabla w; w \in W^{1,2}(S, \mathbb{R}^2)\}$. When S is strictly convex, rotationally symmetric or developable without flat regions, it has been proved in [13,24] that $\mathcal{B} = L^2(S, \mathbb{R}_{\text{sym}}^{2 \times 2})$, i.e. it contains all symmetric matrix fields on S with square integrable entries.

Finally, in (3.3), the quadratic forms

$$\mathcal{Q}_3(F) = D^2 W(\text{Id})(F, F) \quad \text{and} \quad \mathcal{Q}_2(x, F_{\text{tan}}) = \min\{\mathcal{Q}_3(\tilde{F}); \tilde{F} \in \mathbb{R}^{3 \times 3}, (\tilde{F} - F)_{\text{tan}} = 0\}, \quad (3.6)$$

where the form \mathcal{Q}_3 is defined for all $F \in \mathbb{R}^{3 \times 3}$, while $\mathcal{Q}_2(x, \cdot)$ for a given $x \in S$ is defined on tangential minors F_{tan} of such matrices. Both forms \mathcal{Q}_3 and all $\mathcal{Q}_2(x, \cdot)$ are non-negative definite and depend only on the symmetric parts of their arguments.

We now have the following results, stating in particular that the functional \mathcal{I}_4 is the Γ -limit [25] of the scaled energies $h^{-4} \mathcal{I}^h$:

Theorem 3.1. *Let a sequence of deformations $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ satisfy $\mathcal{I}^h(u^h) \leq Ch^4$. Then there exists proper rotations $\tilde{R}^h \in SO(3)$ and translations $c^h \in \mathbb{R}^3$ such that for the renormalized deformations*

$$y^h(x + tn(x)) = (\tilde{R}^h)^T u^h \left(x + t \frac{h}{h_0} \mathbf{n} \right) - c^h : S^{h_0} \rightarrow \mathbb{R}^3$$

defined on the common thin shell S^{h_0} , the following holds.

- (i) y^h converge in $W^{1,2}(S^{h_0}, \mathbb{R}^3)$ to π .
- (ii) The scaled displacements

$$V^h(x) = h^{-1} \int_{-h_0/2}^{h_0/2} y^h(x + tn) - x \, dt \quad (3.7)$$

converge (up to a subsequence) in $W^{1,2}(S, \mathbb{R}^3)$ to some $V \in \mathcal{V}$.

- (iii) The scaled averaged strains

$$B^h(x) = h^{-1} \text{sym } \nabla V^h(x) \quad (3.8)$$

converge (up to a subsequence) weakly in $L^2(S, \mathbb{R}^{2 \times 2})$ to a limit $B \in \mathcal{B}$.

(iv) The lower bound holds

$$\liminf_{h \rightarrow 0} h^{-4} I^h(u^h) \geq \mathcal{I}_4(V, B).$$

Theorem 3.2. For every couple $V \in \mathcal{V}$ and $B \in \mathcal{B}$, there exists a sequence of deformations $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ such that:

- (i) The rescaled sequence $y^h(x + t\mathbf{n}) = u^h(x + t(h/h_0)\mathbf{n})$ converges in $W^{1,2}(S^{h_0}, \mathbb{R}^3)$ to π .
- (ii) The displacements v^h as in (3.7) converge in $W^{1,2}(S, \mathbb{R}^3)$ to V .
- (iii) The strains B^h as in (3.8) converge in $W^{1,2}(S, \mathbb{R}^{2 \times 2})$ to B .
- (iv) There holds

$$\lim_{h \rightarrow 0} h^{-4} I^h(u^h) = \mathcal{I}_4(V, B).$$

The proofs follow through a combination of arguments in [10,13], which we do not repeat here but instead comment on the functional (3.3) and its relationship with the prestrained von Kármán equations for plates.

Here, in analogy with the theory for flat plates $S \subset \mathbb{R}^2$ with incompatible strains [10], in (3.1) we have assumed that the target metric is second order in thickness h for the in-plane stretching (sym ϵ_g), and first order in h for bending (sym κ_g). Owing to this particular choice of scalings, the limit energy \mathcal{I}_4 is composed of exactly two terms, corresponding to stretching and bending. The argument of the integrand in the first term, namely $B - 1/2(A^2)_{\text{tan}} - (\text{sym } \epsilon_g)_{\text{tan}}$, represents the difference of the second-order stretching induced by the deformation $v^h = \text{id} + hV + h^2w^h$ from the target stretching (sym ϵ_g), with $V \in \mathcal{V}$ and $\text{sym } \nabla w^h \rightarrow B$. The argument of the integrand in the second term $(\nabla(A\mathbf{n}) - A\Pi)_{\text{tan}} - (\text{sym } \kappa_g)_{\text{tan}}$, represents the difference of the first-order bending induced by v^h from the target bending (sym κ_g).

In general, the second-order displacement w can be very oscillatory. Owing to the non-trivial geometry of the mid-surface S , the finite strain space \mathcal{B} is usually large and hence a bound on the L^2 norm of the symmetric gradients $\text{sym } \nabla w^h$ implies only a very weak bound on w^h . The limiting tensor B can hence be written only as the symmetric gradient of a very weakly regular distribution (not a classical higher order displacement).

Remark 3.3. When the mid-surface S is elliptic, then for any first-order isometry $V \in \mathcal{V}$, there exists $B \in \mathcal{B} = L^2(S, \mathbb{R}_{\text{sym}}^{2 \times 2})$ such that $B - 1/2(A^2)_{\text{tan}} - (\text{sym } \epsilon_g)_{\text{tan}} = 0$ [14]. This implies that for any V there exists a higher order modification w^h for which in the limit, the second-order target stretching is realized. Thus, the energy \mathcal{I}_4 reduces to

$$\mathcal{I}_4(V) = \frac{1}{24} \int_S \mathcal{Q}_2(x, (\nabla(A\mathbf{n}) - A\Pi)_{\text{tan}} - (\text{sym } \kappa_g)_{\text{tan}}) \, dx,$$

i.e. the bending term which is to be minimized over the space \mathcal{V} . Note that this variational problem is convex (minimizing a convex integral over a linear space \mathcal{V}), and hence it admits only one solution (up to rigid motions). Following the analysis in [14], we see that for elliptic surfaces, all limiting theories for $h^{-\beta} I^h$ under the energy scaling $\beta > 2$, coincide with the linear theory \mathcal{I}_4 as above, while the sublinear theory, to be used in the description of buckling, is the Kirchhoff-like (nonlinear bending) theory corresponding to $\beta = 2$ and derived in [17].

4. The prestrained shallow shell with a first-order isometry constraint:

$$0 < \alpha < 1$$

When the parameter $0 < \alpha < 1$, the highest order terms (of order $h^{2\alpha}$) in the prestrain metric p^h on $(S_\gamma)^h$ pulled back on the flat reference configuration Ω^h , turn out to be ‘compatible’, i.e. entirely generated by the reference displacement $h^\alpha v_0$. In other words, the shallow shell will easily compensate for these terms by rigidly keeping its structure at the h^α order and only will make adjustments at higher orders to the prestrain induced by ϵ_g and κ_g . In the limit as $h \rightarrow 0$

we therefore expect that the effective energy functional on Ω will depend only on the out-of-plane and the in-plane displacements of respective orders h and h^2 . Yet, as we shall see below, the residual curvature of mid-surfaces will appear in a twofold manner: as a linearized first-order isometry constraint on the out-of-plane displacement (4.3) and also as a defining constraint on the space of admissible in-plane displacements. The mid-plate Ω will inherit the space of the first-order infinitesimal isometries (3.4) and the finite strain space (3.5), in the asymptotic limit of vanishing curvature shells.

The space of finite strains $\mathcal{B}_{v_0} \subset L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})$ is defined as

$$\mathcal{B}_{v_0} = \left\{ L^2 - \lim_{\epsilon \rightarrow 0} (\text{sym } \nabla w^\epsilon + \text{sym}(\nabla v^\epsilon \otimes \nabla v_0)); w^\epsilon \in W^{1,2}(\Omega, \mathbb{R}^2), v^\epsilon \in W^{1,2}(\Omega, \mathbb{R}) \right\}.$$

We now identify \mathcal{B}_{v_0} with each of the finite strain spaces of the shallow surfaces S_γ .

Lemma 4.1. *Let the surfaces S_γ be as in (2.1). Then for all $\gamma \neq 0$, the finite strain spaces*

$$\mathcal{B}^\gamma = \left\{ L^2 - \lim_{\epsilon \rightarrow 0} \text{sym } \nabla w^\epsilon; w^\epsilon \in W^{1,2}(S_\gamma, \mathbb{R}^3) \right\},$$

are each isomorphic to \mathcal{B}_{v_0} via the linear isomorphism:

$$\mathcal{T}^\gamma : L^2(S_\gamma, \mathcal{L}_{\text{sym}}^2(TS_\gamma, \mathbb{R})) \rightarrow L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2}).$$

Here, $L^2(S_\gamma, \mathcal{L}_{\text{sym}}^2(TS_\gamma, \mathbb{R}))$ is the space of all L^2 -sections of the bundle of symmetric bilinear forms on S_γ , and \mathcal{T}^γ is naturally defined by

$$[\mathcal{T}^\gamma(\sigma)(x)]_{ij} = \sigma(\phi_\gamma(x))(\partial_i \phi_\gamma(x), \partial_j \phi_\gamma(x)) \quad \forall \text{ a.e. } x \in \Omega \quad \forall \sigma \in L^2(S_\gamma, \mathcal{L}_{\text{sym}}^2(TS_\gamma, \mathbb{R})).$$

Proof. Let $w \in W^{1,2}(S_\gamma, \mathbb{R}^3)$ and write $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) = w \circ \phi_\gamma \in W^{1,2}(\Omega, \mathbb{R}^3)$. Then, for $i, j = 1, 2$ we have

$$(\text{sym } \nabla w)(\partial_i \phi_\gamma, \partial_j \phi_\gamma) = \frac{1}{2}(\partial_i \tilde{w} \cdot \partial_j \phi_\gamma + \partial_j \tilde{w} \cdot \partial_i \phi_\gamma) = [\text{sym } \nabla(\tilde{w}_1, \tilde{w}_2) + \gamma \text{sym}(\nabla \tilde{w}_3 \otimes \nabla v_0)]_{ij}.$$

Take now a sequence $w^\epsilon \in W^{1,2}(S_\gamma, \mathbb{R}^3)$ such that $\lim_{\epsilon \rightarrow 0} \text{sym } \nabla w^\epsilon = B_\gamma \in \mathcal{B}^\gamma$. Then

$$\mathcal{T}^\gamma(B_\gamma) = \lim_{\epsilon \rightarrow 0} \mathcal{T}^\gamma(\text{sym } \nabla w^\epsilon) = \lim_{\epsilon \rightarrow 0} (\text{sym } \nabla(\tilde{w}_1^\epsilon, \tilde{w}_2^\epsilon) + \text{sym}(\nabla(\gamma \tilde{w}_3^\epsilon \otimes \nabla v_0))) \in \mathcal{B}_{v_0},$$

which proves the claim. ■

The following is a consequence of lemma 4.1 [13, lemma 5.6] and [24, lemma 3.3]:

Corollary 4.2. *Assume that*

- (i) either: $v_0 \in \mathcal{C}^{2,1}(\Omega) \cap \mathcal{C}^{1,1}(\bar{\Omega})$ and $\det \nabla^2 v_0 \geq c > 0$ in Ω ,
- (ii) or: $v_0 \in \mathcal{C}^2(\bar{\Omega})$ with $\det \nabla^2 v_0 = 0$ in Ω , and $\nabla^2 v_0$ does not vanish identically on any open region in Ω .

Then:

$$\mathcal{B}_{v_0} = L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2}). \quad (4.1)$$

Indeed, in the study of Lewicka *et al.* [14] we proved that for any strictly elliptic surface S , its finite strain space \mathcal{B} equals $L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})$. As every S_γ is strictly elliptic under assumption (i), the result follows by the equivalence of spaces \mathcal{B}^γ and \mathcal{B}_{v_0} in lemma 4.1. The same observation can be derived directly, as follows. Given $B : \Omega \rightarrow \mathbb{R}_{\text{sym}}^{2 \times 2}$ smooth enough, we first solve for v in

$$\left. \begin{aligned} \text{cof } \nabla^2 v_0 : \nabla^2 v &= -\text{curl}^T \text{curl } B \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned} \right\} \quad (4.2)$$

Then we have

$$\text{curl}^T \text{curl } B = -\text{cof } \nabla^2 v_0 : \nabla^2 v = \text{curl}^T \text{curl}(\nabla v \otimes \nabla v_0) = \text{curl}^T \text{curl}(\text{sym}(\nabla v \otimes \nabla v_0))$$

(see also remark 4.7), and therefore

$$B = \text{sym } \nabla(v_1, v_2) + \text{sym}(\nabla v \otimes \nabla v_0),$$

for some in-plane displacement $(v_1, v_2) : \Omega \rightarrow \mathbb{R}^2$. The density of smooth fields B in the space $L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})$ now yields the result.

Remark 4.3. We expect that property (4.1) is satisfied for a generic v_0 , whenever $\nabla^2 v_0$ does not vanish identically on any open region of Ω . The argument requires studying very weak solutions of the mixed-type equation (4.2). When this equation is degenerate ($v_0 \equiv 0$), \mathcal{B}_{v_0} coincides with the space of all matrix fields in the kernel of the operator $\text{curl}^T \text{curl}$ and hence it is only a proper subset of $L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})$, consisting of symmetric gradients.

We now present the main Γ -convergence result for the shallow shell regime $0 < \alpha < 1$. The proofs which consist of tedious modifications of the arguments in [10,13] are outlined in the electronic supplementary material, appendix.

Theorem 4.4. Let $0 < \alpha < 1$. Assume $u^h \in W^{1,2}((S_{I_{\text{tr}}}^h)^h, \mathbb{R}^3)$ satisfies $I^{h^\alpha, h}(u^h) \leq Ch^4$, where $I^{h^\alpha, h}$ is given as in (2.5). Then there exists $\bar{R}^h \in SO(3)$ and $c^h \in \mathbb{R}^3$ such that for the normalized deformations

$$y^h(x, t) = (\bar{R}^h)^T (u^h \circ \tilde{\phi}_{I_{\text{tr}}^\alpha})(x, ht) - c^h : \Omega^1 \rightarrow \mathbb{R}^3$$

with ϕ_γ and $\gamma = h^\alpha$ as in (2.1), we have

(i) $y^h(x, t)$ converge in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to x .

(ii) The scaled displacements $V^h(x) = h^{-1} \int_{-1/2}^{1/2} y^h(x, t) - x - h^\alpha v_0(x) e_3 dt$ converge (up to a subsequence) in $W^{1,2}(\Omega, \mathbb{R}^3)$ to $(0, 0, v)^T$ where $v \in W^{2,2}(\Omega, \mathbb{R})$ and

$$\text{cof } \nabla^2 v_0 : \nabla^2 v = 0 \quad \text{in } \Omega. \quad (4.3)$$

(iii) The scaled strains

$$B_h = \frac{1}{h} (\text{sym } \nabla(V_1^h, V_2^h) + h^\alpha \text{sym}(\nabla V_3^h \otimes \nabla v_0))$$

converge (up to a subsequence) weakly in L^2 to some $B \in \mathcal{B}_{v_0}$.

(iv) Moreover: $\liminf_{h \rightarrow 0} h^{-4} I^{h^\alpha, h}(u^h) \geq \mathcal{I}_4^\infty(v, B)$, where

$$\begin{aligned} \mathcal{I}_4^\infty(v, B) &= \int_{\Omega} \mathcal{Q}_2 \left(B + \frac{1}{2} \nabla v \otimes \nabla v - (\text{sym } \epsilon_g)_{\text{tan}} \right) \\ &\quad + \frac{1}{24} \int_{\Omega} \mathcal{Q}_2 \left(\nabla^2 v + (\text{sym } \kappa_g)_{\text{tan}} \right), \end{aligned} \quad (4.4)$$

with \mathcal{Q}_2 defined in (3.6).

Theorem 4.5. Let $0 < \alpha < 1$. For every $v \in W^{2,2}(\Omega, \mathbb{R})$ satisfying (4.3) and every $B \in \mathcal{B}_{v_0}$, there exists a sequence of deformations $u^h \in W^{1,2}((S_{I_{\text{tr}}}^h)^h, \mathbb{R}^3)$ such that

(i) The sequence $y^h(x, t) = u^h(x + h^\alpha v_0(x) e_3 + ht n^\nu(x))$ converges in $W^{1,2}(\Omega^1)$ to x .

(ii) The scaled displacements V^h as in (ii) theorem 4.4 converge in $W^{1,2}$ to $(0, 0, v)$.

(iii) The scaled strains B^h as in (iii) theorem 4.4 converge weakly in L^2 to B .

(iv) $\lim_{h \rightarrow 0} h^{-4} I^{h^\alpha, h}(u^h) = \mathcal{I}_4^\infty(v, B)$.

In the special cases of corollary 4.2, we have

Theorem 4.6. Assume additionally that v_0 is such that (4.1) holds. Then, for every $v \in W^{2,2}(\Omega, \mathbb{R})$ satisfying (4.3), there exists a sequence $u^h \in W^{1,2}((S_{I_{\text{tr}}}^h)^h, \mathbb{R}^3)$ such that (i) and (ii) of theorem 4.5 hold, and moreover

$$\lim_{h \rightarrow 0} h^{-4} I^{h^\alpha, h}(u^h) = \frac{1}{24} \int_{\Omega} \mathcal{Q}_2(\nabla^2 v + (\text{sym } \kappa_g)_{\text{tan}}).$$

Remark 4.7. Comparing functionals (4.4) with (3.3), note that the space $\mathcal{V}(S_\gamma)$ of the first-order infinitesimal isometries on S_γ is made of displacements $V : S_\gamma \rightarrow \mathbb{R}^3$ of the form

$$\left. \begin{aligned} V(\phi_\gamma(x)) &= (\gamma v_1(x), h^\alpha v_2(x), v_3) \quad \forall x \in \Omega, \\ \text{such that } (v_1, v_2, v_3) &\in W^{2,2}(\Omega, \mathbb{R}^3) \text{ and } \text{sym } \nabla(v_1, v_2) + \text{sym}(\nabla v_3 \otimes \nabla v_0) = 0. \end{aligned} \right\} \quad (4.5)$$

Indeed, similarly as in the proof of lemma 4.1, the condition $\text{sym} \nabla V = 0$ on S_γ becomes

$$0 = \frac{1}{2}(\partial_i(V \circ \phi_\gamma) \cdot \partial_j \phi_\gamma + \partial_j(V \circ \phi_\gamma) \cdot \partial_i \phi_\gamma) = \text{sym}[\nabla(v_1, v_2) + \nabla v_3 \otimes \nabla v_0]_{ij}.$$

We also see that v_3 can be completed by (v_1, v_2) to $V \in \mathcal{V}_1(S_h)$ as in (4.5) only if

$$\text{cof } \nabla^2 v_0 : \nabla^2 v_3 = 0, \quad (4.6)$$

the latter being also a sufficient condition when Ω is simply connected. This follows from

$$\begin{aligned} \text{curl}^T \text{curl}(\text{sym}(\nabla v_3 \otimes \nabla v_0)) &= \text{curl}^T \text{curl}(\nabla v_3 \otimes \nabla v_0) \\ &= \partial_{22}(\partial_1 v_3 \cdot \partial_1 v_0) + \partial_{11}(\partial_2 v_3 \cdot \partial_2 v_0) - \partial_{12}(\partial_1 v_3 \cdot \partial_2 v_0 + \partial_2 v_3 \cdot \partial_1 v_0) \\ &= -(\partial_{11} v_3 \cdot \partial_{22} v_0 + \partial_{22} v_3 \cdot \partial_{11} v_0 - 2\partial_{12} v_3 \cdot \partial_{12} v_0) = -\text{cof } \nabla^2 v_0 : \nabla^2 v_3. \end{aligned}$$

Hence, the admissible out-of-plane displacements v_3 relevant in (3.3), must obey for the least constraint (4.6), which appears in the 2-scale limiting theory (4.4) as constraint (4.3). This is in contrast with the unconstrained 2-scale limiting theory (5.3) developed in §5.

Remark 4.8. To put the last two results in another context, we draw the reader's attention to the forthcoming paper [22], where we analyse the Γ -limit of the shallow shell energies $(1/h^{2\alpha+2})I^{h^\alpha, h}$ on shells with curvature of order h^α . This energy scaling is produced by forces of appropriate magnitude or by prestrains of a different order than those considered in this paper. Our main result in [22] concerns the case $\alpha < 1$, where we can establish that in the special case $\det \nabla^2 v_0 \equiv c_0 > 0$, the Γ -limit is a linearized Kirchhoff model with a Monge–Ampère curvature constraint

$$\det \nabla^2 v = \det \nabla^2 v_0 \quad (4.7)$$

on the admissible out-of-plane displacements $v \in W^{2,2}(\Omega)$. Constraint (4.3) can be interpreted as a linearization of (4.7), thereby highlighting the relationship between the two models for elliptic shallow shells.

5. The generalized Donnell–Mushtari–Vlasov model for a prestrained shallow shell: $\alpha = 1$

When the parameter $\alpha = 1$, i.e. the curvature of the mid-surface covaries with the thickness, so that $\gamma = h$. For small h , the growth tensors on $(S_h)^h$ are then defined by (2.6) and the corresponding metric $p^h = (q^h)^T q^h$ is given by

$$p^h(\phi_h(x) + x_3 n^h(x)) = \text{Id} + 2h^2 \text{sym } \epsilon_g(x) + 2hx_3 \text{sym } \kappa_g(x) + \mathcal{O}(h^3).$$

Let $v^h = u^h \circ \tilde{\phi}_h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$, via diffeomorphisms $\tilde{\phi}_h$ in (2.3). By this simple change of variables, we see that

$$\begin{aligned} I^{h,h}(u^h) &= \frac{1}{h} \int_{(S_h)^h} W(\nabla u^h (q^h)^{-1}) \\ &= \frac{1}{h} \int_{\Omega^h} W((\nabla v^h)(\nabla \tilde{\phi}_h)^{-1} (q^h \circ \tilde{\phi}_h)^{-1}) \cdot \det \nabla \tilde{\phi}_h \, d(x, x_3) \\ &= \frac{1}{h} \int_{\Omega^h} W((\nabla v^h)(b^h)^{-1}) \cdot \det \nabla \tilde{\phi}_h \, d(x, x_3), \end{aligned}$$

where

$$b^h = (q^h \circ \tilde{\phi}_h) \nabla \tilde{\phi}_h.$$

In order to understand the structure of b^h , we need the following result:

Lemma 5.1. *The pull-back of the metric p^h through $\tilde{\phi}_h$ satisfies*

$$\begin{aligned} \forall (x, x_3) \in \Omega^h \quad g^h(x, x_3) &= (\nabla \tilde{\phi}_h)^T (p^h \circ \tilde{\phi}_h) (\nabla \tilde{\phi}_h) \\ &= \text{Id} + h^2 (2 \text{sym } \epsilon_g(x) + (\nabla v_0(x) \otimes \nabla v_0(x))^*) \\ &\quad + 2hx_3 (\text{sym } \kappa_g(x) - (\nabla^2 v_0(x))^*) + \mathcal{O}(h^3), \end{aligned}$$

where $F^* \in \mathbb{R}^{3 \times 3}$ denotes the matrix whose only non-zero entries are in its 2×2 principal minor given by $F \in \mathbb{R}^{2 \times 2}$.

Proof. By a direct calculation, we obtain

$$\partial_1 \tilde{\phi}_h = (1 - x_3 h \partial_{11}^2 v_0, -x_3 h \partial_{12}^2 v_0, h \partial_1 v_0) + \mathcal{O}(h^3),$$

$$\partial_2 \tilde{\phi}_h = (-x_3 h \partial_{12}^2 v_0, 1 - x_3 h \partial_{22}^2 v_0, h \partial_2 v_0) + \mathcal{O}(h^3)$$

and

$$\partial_3 \tilde{\phi}_h = \mathbf{n}^h = \left(-h \partial_1 v_0, -h \partial_2 v_0, 1 - \frac{1}{2} h^2 |\nabla v_0|^2 \right) + \mathcal{O}(h^3).$$

Hence

$$(\nabla \tilde{\phi}_h)^T (\nabla \tilde{\phi}_h) = \text{Id}_3 - 2x_3 h (\nabla^2 v_0)^* + h^2 (\nabla v_0 \otimes \nabla v_0)^* + \mathcal{O}(h^3)$$

$$(\nabla \tilde{\phi}_h)^T (2h^2 \text{sym } \epsilon_g + 2hx_3 \text{sym } \kappa_g) (\nabla \tilde{\phi}_h) = 2h^2 \text{sym } \epsilon_g + 2hx_3 \text{sym } \kappa_g + \mathcal{O}(h^3),$$

in view of $\nabla \tilde{\phi}_h = \text{Id}_3 + \mathcal{O}(h)$, and the result follows. ■

Note that $(b^h)^T b^h = g^h$ and therefore by the polar decomposition of matrices

$$b^h = R(x, x_3) a^h \quad \text{on } \Omega^h$$

for some $R(x, x_3) \in \text{SO}(3)$ and the symmetric growth tensor a^h given by

$$a^h = \sqrt{g^h} = \text{Id} + h^2 \left(\text{sym } \epsilon_g + \frac{1}{2} (\nabla v_0 \otimes \nabla v_0)^* \right) + hx_3 (\text{sym } \kappa_g - (\nabla^2 v_0)^*) + \mathcal{O}(h^3). \quad (5.1)$$

For isotropic W , it directly follows that

$$\begin{aligned} I^{h,h}(u^h) &= \frac{1}{h} \int_{\Omega^h} W((\nabla v^h)(a^h)^{-1} R(x)^{-1}) \cdot \det \nabla \tilde{\phi}_h \, d(x, x_3) \\ &= \frac{1}{h} \int_{\Omega^h} W((\nabla v^h)(a^h)^{-1}) \cdot (1 + \mathcal{O}(h)) \, d(x, x_3). \end{aligned} \quad (5.2)$$

Heuristically, modulo the change of variable $\tilde{\phi}_h$ the problem reduces then to the study of deformations of the flat thin film Ω^h with the prestrain a^h . Indeed, by exactly the same analysis as in [10, theorems 1.2 and 1.3], we obtain in the general (not necessarily isotropic) case, the following result:

Theorem 5.2. *Assume that $u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3)$ satisfies $I^{h,h}(u^h) \leq Ch^4$. Then there exists proper rotations $\tilde{R}^h \in \text{SO}(3)$ and translations $c^h \in \mathbb{R}^3$ such that for the normalized deformations*

$$y^h(x, t) = (\tilde{R}^h)^T (u^h \circ \tilde{\phi}_h)(x, ht) - c^h : \Omega^1 \longrightarrow \mathbb{R}^3$$

defined by means of (2.3) on the common domain $\Omega^1 = \Omega \times (-1/2, 1/2)$ the following holds:

- (i) $y^h(x, t)$ converge in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to x .
- (ii) The scaled displacements $V^h(x) = h^{-1} \int_{-1/2}^{1/2} y^h(x, t) - x \, dt$ converge (up to a subsequence) in $W^{1,2}(\Omega, \mathbb{R}^3)$ to the vector field of the form $(0, 0, v)^T$ and $v \in W^{2,2}(\Omega, \mathbb{R})$.

- (iii) The scaled in-plane displacements $h^{-1}V_{\tan}^h$ converge (up to a subsequence) weakly in $W^{1,2}$ to $w \in W^{1,2}(\Omega, \mathbb{R}^2)$.
- (iv) Moreover: $\liminf_{h \rightarrow 0} h^{-4}I^{h,h}(u^h) \geq \mathcal{I}_4^1(w, v)$ where

$$\begin{aligned} \mathcal{I}_4^1(w, v) = & \frac{1}{2} \int_{\Omega} \mathcal{Q}_2 \left(\text{sym } \nabla w + \frac{1}{2} \nabla v \otimes \nabla v - \frac{1}{2} \nabla v_0 \otimes \nabla v_0 - (\text{sym } \epsilon_g)_{\tan} \right) \\ & + \frac{1}{24} \int_{\Omega} \mathcal{Q}_2(\nabla^2 v - \nabla^2 v_0 + (\text{sym } \kappa_g)_{\tan}). \end{aligned} \quad (5.3)$$

In the same manner, applying the proof of [10, theorem 1.4] to (5.2), yields:

Theorem 5.3. For every $v \in W^{2,2}(\Omega, \mathbb{R})$ and $w \in W^{1,2}(\Omega, \mathbb{R}^2)$, there exists a sequence of deformations $u^h \in W^{1,2}((S_h)^h, \mathbb{R}^3)$ such that

- (i) The sequence $y^h(x, t) = u^h(x + hv_0(x)e_3 + htn^h(x))$ converges in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to x .
- (ii) The displacements V^h as in (ii) theorem 5.2 converge in $W^{1,2}$ to $(0, 0, v)$.
- (iii) The in-plane displacements $h^{-1}V_{\tan}^h$ converge in $W^{1,2}$ to w .
- (iv) $\lim_{h \rightarrow 0} h^{-4}I^{h,h}(u^h) = \mathcal{I}_{g,v_0}(w, v)$.

6. The prestrained plate model and the Euler–Lagrange equations: $\alpha > 1$

When the parameter $\alpha > 1$, we calculate the pull-back of the induced metric $p^h = (q^h)^T q^h$, to the flat plate Ω^h , via the change of variable ϕ_γ as in (2.3). Just as in lemma 5.1, we obtain

$$\begin{aligned} g^h = (\tilde{\phi}_{h^\alpha})^* p^h = & \text{Id}_3 + h^{2\alpha}(\nabla v_0 \otimes \nabla v_0)^* - 2h^\alpha x_3(\nabla^2 v_0)^* \\ & + 2h^2 \text{sym } \epsilon_g + 2hx_3 \text{sym } \kappa_g + \mathcal{O}(h^3). \end{aligned} \quad (6.1)$$

It is therefore clear that the prestrain terms (ϵ_g, κ_g) take over the effect of shallowness and hence the limiting theory in the scaling regime h^4 is that derived in [10], coinciding with results of theorems 5.2 and 4.4 for the case $v_0 = 0$ and with the results of theorem 3.1 for $S \subset \mathbb{R}^2$

$$\text{and } \left. \begin{aligned} \forall v \in W^{2,2}(\Omega, \mathbb{R}) \quad \forall w \in W^{1,2}(\Omega, \mathbb{R}^2) \\ \mathcal{I}_4^0(w, v) = \frac{1}{2} \int_{\Omega} \mathcal{Q}_2 \left(\text{sym } \nabla w + \frac{1}{2} \nabla v \otimes \nabla v - (\text{sym } \epsilon_g)_{\tan} \right) \\ + \frac{1}{24} \int_{\Omega} \mathcal{Q}_2(\nabla^2 v + (\text{sym } \kappa_g)_{\tan}). \end{aligned} \right\} \quad (6.2)$$

Indeed, consider the prestrained von Kármán shell model \mathcal{I}_4 discussed in §3 for a degenerate situation $S \subset \mathbb{R}^2$. The term $B - 1/2(A^2)_{\tan}$ reduces to $1/2(\nabla w + (\nabla w)^T + \nabla v \otimes \nabla v)$, where w and $v = V^3$ are, respectively, the in-plane and the out-of-plane displacements of S . The term $(\nabla(A\mathbf{n}) - A\mathbf{T})_{\tan}$ reduces also to $-\nabla^2 v$. Therefore, when $S \subset \mathbb{R}^2$, \mathcal{I}_4 coincides with the model \mathcal{I}_4^0 and with the models \mathcal{I}_4^∞ and \mathcal{I}_4^1 in the degenerate case $v_0 = 0$.

Remark 6.1. We point out a qualitative difference between the out-of-plane displacements v in the argument of \mathcal{I}_4^0 and \mathcal{I}_4^1 and those appearing as the arguments of \mathcal{I}_4^∞ . The former are the net lowest order out of plane displacements of the limit deformations which are of order h , as suggested by theorem 5.2 (ii), but, according to theorem 4.4 (ii), when $\alpha < 1$, the latter are the second highest order term of the expansion of the deformation after $h^\alpha v_0$. Hence, one should replace v in (5.3) or (6.5) through a change of variables by $v + h^{\alpha-1}v_0$ in order to quantitatively compare this model with the variational model \mathcal{I}_4^∞ in (4.4).

As shown in [10], under the assumption of W being isotropic, the Euler–Lagrange equations of \mathcal{I}_4 under this degeneracy condition (or equivalently the Euler–Lagrange equations of \mathcal{I}_4^0) can be then written in terms of the displacement v and the Airy stress potential Φ

$$\text{and} \quad \left. \begin{aligned} \Delta^2 \Phi &= -Y(\det \nabla^2 v + \lambda_g) \\ Z \Delta^2 v &= [v, \Phi] - Z \Omega_g, \end{aligned} \right\} \quad (6.3)$$

where Y is the Young modulus, Z the bending stiffness, ν the Poisson ratio (given in terms of the Lamé constants μ and λ), and

$$\text{and} \quad \left. \begin{aligned} \lambda_g &= \text{curl}^T \text{curl}(\epsilon_g)_{2 \times 2} = \partial_{22}(\epsilon_g)_{11} + \partial_{11}(\epsilon_g)_{22} - \partial_{12}((\epsilon_g)_{12} + (\epsilon_g)_{21}) \\ \Omega_g &= \text{div}^T \text{div}((\kappa_g)_{2 \times 2} + \nu \text{cof}(\kappa_g)_{2 \times 2}) \\ &= \partial_{11}((\kappa_g)_{11} + \nu(\kappa_g)_{22}) + \partial_{22}((\kappa_g)_{22} + \nu(\kappa_g)_{11}) + (1 - \nu)\partial_{12}((\kappa_g)_{12} + (\kappa_g)_{21}). \end{aligned} \right\} \quad (6.4)$$

Equations (6.3), generalizing the von Kármán equations [26], are based on a thermoelastic analogy to growth [7,8] and can also be derived using a formal perturbation theory [9].

On the other hand, the following system was introduced in [20], as a mathematical model of blooming activated by differential lateral growth from an initial non-zero transverse displacement field v_0

$$\text{and} \quad \left. \begin{aligned} \Delta^2 \Phi &= -Y(\det \nabla^2 v - \det \nabla^2 v_0 + \lambda_g) \\ Z(\Delta^2 v - \Delta^2 v_0) &= [v, \Phi] - Z \Omega_g, \end{aligned} \right\} \quad (6.5)$$

A similar calculation as in [10] then shows that (6.5) can be viewed as the Euler–Lagrange equations corresponding to the energy functional \mathcal{I}_4^1 . We will now show that (6.5) can be directly derived from equations (6.3).

Proposition 6.2. *System (6.5) can be derived from equations (6.3) by pulling back the prestrain tensors ϵ_g and κ_g from a sequence of shallow shells $(S_h)^h$ generated by the vanishing out-of-plane displacements $h v_0$.*

Proof. By lemma 5.1 we see that the growth tensor on Ω^h is given by (5.1). Applying (6.4) to the modified strain and curvature in a^h , to the leading order, we obtain

$$\begin{aligned} \lambda_g(v_0) &= \text{curl}^T \text{curl} \left((\text{sym } \epsilon_g)_{\text{tan}} + \frac{1}{2} \nabla v_0 \otimes \nabla v_0 \right) = \lambda_g + \det \nabla^2 v_0 \\ \Omega_g(v_0) &= \text{div}^T \text{div} \left(((\text{sym } \kappa_g)_{\text{tan}} - \nabla^2 v_0) + \nu \text{cof}((\text{sym } \kappa_g)_{\text{tan}} - \nabla^2 v_0) \right) \\ &= \Omega_g - \Delta^2 v_0, \end{aligned}$$

where the last equality follows from $\text{div cof } \nabla^2 v_0 = 0$. Consequently, (6.3) for the growth tensor (5.1) becomes exactly (6.5). ■

7. The energy scaling

A straightforward consequence of our results is the following assertion about the scaling of the infimum elastic energies of the thin prestrained shallow shells in the von Kármán regime (2.6).

Theorem 7.1. *Let $\alpha > 0$ and let the sequence of thin shells $(S_\gamma)^h$ be given as in (2.2) with the elastic energies of deformations $I^{\gamma,h}$ as in (2.5). Assume that*

$$\text{curl}(\text{sym } \kappa_g)_{\text{tan}} \neq 0 \quad \text{in } \Omega. \quad (7.1)$$

Then, there exists constants $c, C > 0$ for which

$$\forall 0 < h \ll 1 \quad c \leq \inf_{u \in W^{1,2}((S_{h^\alpha})^h, \mathbb{R}^3)} \frac{1}{h^4} I^{h^\alpha, h}(u) \leq C. \quad (7.2)$$

Indeed, the condition $\text{curl}(\text{sym } \kappa_g)_{\text{tan}} \equiv 0$ is equivalent to $(\text{sym } \kappa_g)_{\text{tan}} = \nabla^2 v$, for some $v : \Omega \rightarrow \mathbb{R}$. If not satisfied, the bending term in (4.4) is always positive, yielding the lower bound in (7.2). The existence of a recovery sequence in theorems 4.5 and 5.3 and [10] implies the upper bound.

Remark 7.2. Incompatibility condition (7.1) can be relaxed depending on the specific value of α , and the assumed energy level, see e.g. [10] for a more involved scaling analysis when $\alpha > 1$. Heuristically, conditions of similar type imply that the Riemann curvature tensor of the induced metric p^h is non-zero and hence, in view of [17, theorem 2.2], they guarantee the positivity of the infimum of $I^{h^\alpha, h}$. In a further step, we observe that, when p^h is close to be flat, the scaling regime depends on the magnitude of the first non-zero term of the expansion of its curvature tensor. Note also that when $\alpha < 1$, the first two non-zero terms after identity in (6.1) have no bearing on the first non-zero terms in the expansion of the curvature. Analogously, the induced prestrains $\kappa'_g = \nabla^2 v_0$ and $\epsilon'_g = 1/2(\nabla^2 v_0 \otimes \nabla^2 v_0)$ corresponding to the scalings h^α and $h^{2\alpha}$ do not satisfy neither conditions (1.13) nor (1.14) of [10]. Therefore, the energy infimum must naturally fall below h^4 , i.e. in the regime $h^{2\alpha+2}$.

8. Discussion

Our analysis has rigorously derived a general theory of shells with residual strain arising from relative growth, inhomogeneous swelling, plasticity, etc. In fact, there are many such theories; each is a consequence of the scalings of the shell curvature relative to the magnitude of the strain incompatibility induced by the in-plane and curvature growth tensors. Indeed, for any exponent $\alpha \geq 0$ we have considered the following energies of deformations on weakly prestrained shallow shells:

$$I^h(u) = \frac{1}{h} \int_{(S_{h^\alpha})^h} W((\nabla u)(q^h)^{-1}) \quad \forall u \in W^{1,2}((S_{h^\alpha})^h, \mathbb{R}^3),$$

with the growth tensor q^h given by (2.6) on thin shells of form (2.2) around the mid-surface

$$S_{h^\alpha} = \phi_{h^\alpha}(\Omega), \quad \phi_{h^\alpha}(x) = (x, h^\alpha v_0(x)), \quad v_0 \in C^{1,1}(\bar{\Omega}, \mathbb{R}).$$

We have established that independent of the value of α , the scaling for the infimum of the energy is always determined by the prestrain and is of order h^4 under our current assumption (7.1).

When $\alpha > 1$, the prestrain overwhelms the role of shallowness so that the limiting theory is the one derived in [10], coinciding with results of theorem 5.2 for the case $v_0 = 0$ and yielding the Euler–Lagrange equations (6.3). When $\alpha = 1$, one recovers the recently postulated model [20], discussed in this paper. For the case $0 < \alpha < 1$, the limiting theory reduces to a new constrained theory and can be viewed as a plate theory where the non-trivial geometric structure of the shallow shell is inherited by the plate, or equivalently it can be considered as the natural limit of the generalized von Kármán theories (3.3) on the shallow midsurface S_γ as $\gamma \rightarrow 0$. This may be contrasted with a similar problem considered by Lewicka *et al.* [22], where the Γ -limit is discussed in the context of the energy scaling as $h^{2\alpha+2}$. The relative ordering of the energy is compatible with the case where the role of shallowness is affected by the relative scaled magnitude of the body forces or prestrains, so that the choice of α has a bearing on the limiting model. Our analysis in this paper and in Lewicka *et al.* [22] is thus the beginning of an exploration that includes many possible scenarios.

A natural generalization of our results would be to allow for different scaling regimes for the growth tensors. Overall, there are three independent parameters: one associated with scaling of the shallowness, and two that characterize the incompatible strains in terms of their dependence on the thickness h in the form h^α . The resulting theories depend on the choice of scalings for these

three parameters. Thus, there is no single *correct* model in general, but specific situations naturally lead to choices of particular scalings for the relative magnitude of the thickness, the shallowness and the differential growth and determines the effective theory, as we have shown here.

Funding statement. This project is based upon work supported by, among others, the National Science Foundation. M.L. is partially supported by the NSF grant nos. DMS-0707275 and DMS-0846996, and by the Polish MN grant no. N N201 547438. L.M. is supported by the MacArthur Foundation, M.R.P. is partially supported by the NSF grant nos. DMS-0907844 and DMS-1210258.

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